XIII.

A New Method for Correcting a Planet's Orbit.

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(*Communicated by Professor Peirce.*)

1. It will be assumed that values of the elements are so nearly known, that the squares of their deviation from the true ones can be neglected; the mass of the planet will also be neglected.

2. The approximate elements will be denoted as follows: —

M° the mean anomaly at the origin of time;

p° the semi-parameter of the orbit;

φ° the angle of eccentricity ($e^\circ = \sin \varphi^\circ$);

π° the longitude of the perihelion;

Ω° the longitude of the ascending node;

i° the inclination;

L° (the mean longitude at the time t_0) $= M^\circ + \pi^\circ$;

a° (the mean distance) $= p^\circ \sec^2 \varphi^\circ$;

μ° (the mean daily motion) $= \frac{k}{\sin 1''} a^{\circ - \frac{3}{2}}$, where $\log k = 8.2355814 - 10$;

$\log k - \log \sin 1'' = 3.5500066$.

The corrected elements to be found will be denoted by the same letters without the $^\circ$; and the prefix δ to any of these letters denotes the correction to be applied to, or any variations of, these same elements.

Analogous modifications of all functions of these quantities will be expressed in the same way.

We shall use the expressions D_M, &c., however, to denote the partial differential coefficients with respect to M, &c. of their functions, even when, for instance, M has the value M°; and so with the rest.

3. But instead of computing *directly* the partial differentials of any coördinates with

respect to the elements, we shall introduce certain intermediate linear functions of the variations δ M, &c., which last may be considered as themselves variations of conceivable finite functions of M, &c.

4. There are, however, two distinct kinds of changes to which the elements are subject; we may (1) vary the magnitude of the orbit and its position in its own plane, or (2) the position of the plane of the orbit. If the latter change alone takes place, it will affect Ω and i, and through Ω the value of π; so that in this case $\delta \pi = \delta \Omega (1 - \cos i)$. But π is also subject to other changes, those of the kind (1); it will therefore be better if we make

$$\delta \chi = \delta \pi - (1 - \cos i) \; \delta \Omega. \tag{1}$$

Putting, for convenience, $\chi^\circ = \pi^\circ$, and substituting $\chi = \chi^\circ + \delta \chi$, $\pi = \pi^\circ + \delta \pi$,

$$\chi = \pi - (1 - \cos i) \; \delta \Omega. \tag{2}$$

We shall, instead of the longitude π of the perihelion, substitute the distance χ of the perihelion, counted from that point, A, fixed in the orbit, which is as far back, in a direction contrary to the motion, from the ascending node, as the equinox is; the former distance being counted upon the orbit and the latter upon the ecliptic.

5. Let now r, v denote the planet's actual radius-vector and true anomaly, at the time t; and (as before stated, see § 2) r°, v° the same, calculated with the approximate elements; and δr, δv the variations $r - r^\circ$, $v - v^\circ$; we shall then get these quantities, as also the planet's heliocentric longitude, $\varDelta$, counted from the point A, just alluded to, in the following way:—

$$\left.\begin{aligned} E^\circ - e^\circ \sin E^\circ &= \mu^\circ t + M^\circ, \\ r^\circ \cos v^\circ &= a^\circ (\cos E^\circ - e^\circ), \\ r^\circ \sin v^\circ &= a^\circ \cos \varphi^\circ \sin E^\circ, \\ \varDelta^\circ &= v^\circ + \pi^\circ. \end{aligned}\right\} \quad (3) \qquad \left.\begin{aligned} E - e \sin E &= \mu t + M, \\ r \cos v &= a (\cos E - e), \\ r \sin v &= a \cos \varphi \sin E, \\ \varDelta &= v + \chi. \end{aligned}\right\} \quad (4)$$

6. Let now z be a new function of the elements and their variations, and of the time t, such that

$$\left.\begin{aligned} \mathfrak{E} - e^\circ \sin \mathfrak{E} &= \mu^\circ z + M^\circ, \\ \mathfrak{r} \cos \mathfrak{v} &= a^\circ (\cos \mathfrak{E} - e^\circ), \\ \mathfrak{r} \sin \mathfrak{v} &= a^\circ \cos \varphi^\circ \sin \mathfrak{E}, \\ \varDelta &= \mathfrak{v} + \pi^\circ, \end{aligned}\right\} \tag{5}$$

We have thus obtained $\varDelta$, the direction in the orbit-plane as seen from the sun, as a function of the approximate elements and z, precisely as it is of the corrected ones and of t; but in order to determine the position in space which the varied elements would give us, we must superpose other variations.

7. The first of these will be $\delta\, w$, such that if $w^\circ = 0$,

$$r = \mathfrak{r}\, c^{w - w^\circ} = \mathfrak{r}\, c^{\delta\, w}, \tag{6}$$

c being the base of the Napierian system of logarithms.

By its aid, and that of z, we can take account of all the variations of the elements which give the dimensions, &c. of the orbit, but not of changes of the orbit-plane itself; and z may be considered as the time when the planet would be in the same heliocentric longitude (modified as before, like π), as it really is, if its elements were M°, &c. The other variation, $\delta\, w$, may be defined as the change necessary to adapt the natural logarithm of r, calculated by means of the approximate elements and for the time z, to its true value for the time t.

8. It will be noticed that we have so far adopted a similar course, in representing the effect of changes in the elements, to that which Mr. Hansen has employed in his theory of perturbations. Our mode of expressing it is taken, with the necessary changes, from Prof. Zech's article on Hansen's Method, Vol. XL. of the *Astronomische Nachrichten*, reproduced by Prof. Encke, *Mathematische Abhandlungen der Berliner Akademie* for 1855, p. 39.

9. The orbit-plane will be made to assume a new position by the change of Ω and i; and the planet will be at a distance δ Z from its position in the unchanged plane; the axis of Z being perpendicular to the orbit.

10. Our first work will be to compute the partial differentials of z, w, Z with respect to the elements. It may be noticed, however, that these quantities, z, w, Z, enjoy but a temporary existence as functions of the elements; they correspond to nothing actually existent. We shall, for convenience, omit the °.

11. By the *Theoria Motus*, article 15, p. 15,

$$d\, v = \frac{a\, a \cos\varphi}{r\, r}\, d\, (\mathrm{M} + \mu\, t) + \frac{a\, a}{r\, r} \frac{(r + p)}{a} \sin \mathrm{E}\, d\, \varphi;$$

Hence,
$$d\, \varLambda = \frac{a\, a \cos\varphi}{r\, r}\, d\, (\mathrm{L} - \pi + \mu\, t) + \frac{a\, a}{r\, r} \frac{r + p}{a} \sin \mathrm{E}\, d\, \varphi + d\, \chi;$$

Letting now $\mathrm{L} - \mathrm{L}_{,} = \pi - \chi$, and $\mathrm{L}_{,} - \mathrm{M} = \chi$, there will result

$$d\, \varLambda = \frac{a\, a \cos\varphi}{r\, r}\, d\, (\mathrm{L}_{,} - \chi + \mu\, t) + \frac{a\, a}{r\, r} \frac{r + p}{a} \sin \mathrm{E}\, d\, \varphi + d\, \chi. \tag{7}$$

From (5) we get
$$\delta\, \varLambda = \delta\, z \frac{d\, \varLambda}{d\, z}; \tag{8}$$

and as $z^\circ = t^\circ$ and $\delta\, z$ is very small, we can assume $\frac{d\, \varLambda}{d\, z} = \frac{d\, \varLambda}{d\, t}$. So will

$$\delta\, \varLambda = \delta\, z \frac{d\, \varLambda}{d\, t}. \tag{9}$$

The law of areas gives us

$$r\,r\,\frac{d\,\Lambda}{d\,t}=k\sqrt{p}=a\,\dot{a}\cos\varphi\,.\,\mu\,, \tag{10}$$

and from (7), (9), (10),

$$\delta\,z=\frac{\delta\,L_{\prime}}{\mu}+\frac{t\,\delta\,\mu}{\mu}+\frac{r+p}{\mu\,a\cos\varphi}\sin\mathrm{E}\,\delta\,\varphi+\frac{1}{\mu}\left(\frac{r\,r}{a\,a\cos\varphi}-1\right)\delta\,\chi. \tag{11}$$

It may be sometimes more convenient to use

$$\mu\,\delta\,z=\delta\,\mathrm{L}_{\prime}+t\,\delta\,\mu+\frac{r+p}{a\cos\varphi}\sin\mathrm{E}\,\delta\,\varphi+\left(\frac{r\,r}{a\,a\cos\varphi}-1\right)\delta\,\chi. \tag{12}$$

12. The equation $\quad p=\left(\frac{k}{\sin 1''}\right)^{\frac{2}{3}}\mu^{-\frac{4}{3}}\cos^2\varphi$

gives $\quad \log p=+\tfrac{2}{3}\log k-\tfrac{2}{3}\log\sin 1''-\tfrac{4}{3}\log\mu+2\log\cos\varphi;$

and therefore, $\quad \tfrac{4}{3}\log\mu=+\tfrac{2}{3}\log k-\tfrac{2}{3}\log\sin 1''-\log p+2\log\cos\varphi;$

$$\frac{d\,\mu}{\mu}=-\tfrac{3}{2}\,d\log p-3\tan\varphi\,d\,\varphi. \tag{13}$$

If, as we propose, we consider μ as a function of $\log p$ and φ, we shall get

$$\mathrm{D}_{\log p}\,z=-\tfrac{3}{2}\,t \tag{14}$$

$$\mathrm{D}_{\varphi}\,z=-3\,t\tan\varphi+\frac{r+p}{\mu\,a\cos\varphi}\sin\mathrm{E}, \tag{15}$$

which will be combined with $\quad \mathrm{D}_{\mathrm{L}_{\prime}}\,z=\frac{1}{\mu}; \qquad (16)$

$$\mathrm{D}_{\chi}\,z=\frac{1}{\mu}\left(\frac{r\,r}{a\,a\cos\varphi}-1\right). \tag{17}$$

13. As is well known, $\quad r=\frac{p}{1+e\cos(\Lambda-\chi)}; \qquad (18)$

but we also have, by (6), $\quad r=\mathfrak{r}\,c^{w},$

$$d\log r=d\log\mathfrak{r}+d\,w\,, \tag{18'}$$

$$d\log\mathfrak{r}=\frac{d\log\mathfrak{r}}{d\,t}\,d\,z,$$

by considerations mentioned in § 11. And as, likewise, we can omit the products *inter se* of the variations of r, among which is $\mathfrak{r}-r$, we shall have, near enough for our present purpose,

$$\frac{d\log\mathfrak{r}}{d\,t}=\frac{d\log r}{d\,t};$$

so that $\quad \delta\,w=\delta\log r-\delta\log\mathfrak{r},$

$$=\delta\log r-\delta\,z\,\frac{d\log r}{d\,t},$$

$$=\delta\log r-\delta\,\Lambda\,\frac{d\log r}{d\,\Lambda}; \tag{18''}$$

And hence, by (18), (18″),

$$\delta w = \delta \log p - \frac{\cos \varphi \cos (A - \chi)}{1 + e \cos (A - \chi)} \delta \varphi - \frac{e \sin (A - \chi)}{1 + e \cos (A - \chi)} \delta \chi ,$$

$$= \delta \log p - \frac{r \cos \varphi \cos v}{p} \delta \varphi - \frac{e r \sin v}{p} \delta \chi ;$$

$$= \delta \log p - \frac{r \cos v}{a \cos \varphi} \delta \varphi - \tan \varphi \sin \mathrm{E}\, \delta \chi ; \tag{19}$$

or,

$$= -\tfrac{2}{3} \frac{\delta \mu}{\mu} - \left(2 \tan \varphi + \frac{r \cos v}{a \cos \varphi}\right) \delta \varphi - \tan \varphi \sin \mathrm{E}\, \delta \chi . \tag{19'}$$

If the planet's place in the corrected orbit be referred to the plane of the uncorrected, it will not be in this plane, but above or below it, if the node and inclination do not remain unchanged. The coördinate δ Z, perpendicular to the plane (A), which, as before stated, is that of the approximate orbit, can be ascertained by the usual method.

Thus, let (counted from the ascending node upon the ecliptic) Θ be the ascending node of the corrected orbit upon the plane (A), and δ I its inclination to that plane. We shall then have $\delta \mathrm{Z} = r \sin \delta \mathrm{I} \sin (v + \omega - \Theta)$, where $\omega = \pi - \Omega$.

But to determine Θ and δ I we shall have, by spherical trigonometry (omitting infinitesimals of the second order),

$$\left.\begin{aligned} \sin \delta \mathrm{I} \sin \Theta &= \sin i \sin \delta \Omega , \\ \sin \delta \mathrm{I} \cos \Theta &= \sin \delta i , \end{aligned}\right\}$$

or,

$$\delta \mathrm{Z} = r \sin \delta i \sin (v + \omega) - r \sin i \sin \delta \Omega \cos (v + \omega). \tag{20}$$

14. So far, we have briefly shown how to resolve the effect upon the planet's place in its orbit of slight variations of the first order in the elements, into three portions: the first, which, divided by its velocity at the time, we have called δz, we have considered as the variation of a function, z, of the time t; differing from t only by variations of the order of the changes in the elements. This is, of course, in the direction of the tangent to the orbit. The second portion we denote by $r \delta w$, and is in the direction of the radius-vector; the third, perpendicular to the plane of the orbit.

The first practical use we shall make of this will be to compute the variations of any geocentric coördinates with respect to the elements; and afterwards study the same relations with respect to certain new geocentric coördinates which we shall adopt.

Goetze, *Ergänzungsheft zu den Astronomische Nachrichten*, Altona, 1849, has shown that if the planet's geocentric place be defined by its position with regard to any plane (II.), and if any other plane (I.) (with regard to which ω_0, Ω_0, i_0, and χ_0 denote what the distance of the perihelion from the ascending node upon the ecliptic, the longitude of that latter point, the inclination, and, lastly, the angle χ, used above, do

with regard to the ecliptic) be employed as a plane of reference for the elements; and again, if $l\ b$ denote, with respect to the other plane (II.), what geocentric longitude and latitude do to the ecliptic; if, yet again, R, S, W denote, with Encke, the directions respectively of the radius-vector, the perpendicular to it in the direction of the motion, and the perpendicular to the orbit; and also F, the direction of the planet as seen from the earth; E, a direction perpendicular to F in a plane perpendicular to (II.), the "longitude," l, of any line in which is the same as that of F; and finally, D, a direction perpendicular to both E and F (the positive direction of E being that of $b = 90°$, and that of D, that of $l + 90°$); we shall have (using, with Encke, R S to denote the angle made between R and S, and so with other letters),

$$\left.\begin{aligned} \Delta \cos b\, \delta l &= \cos \mathrm{R\,D}\, \delta r + r \cos \mathrm{S\,D}\, (\delta v + \delta \chi_0) + \delta \mathrm{Z} \cos \mathrm{W\,D}, \\ \Delta\, \delta b &= \cos \mathrm{R\,E}\, \delta r + r \cos \mathrm{S\,E}\, (\delta v + \delta \chi_0) + \delta \mathrm{Z} \cos \mathrm{W\,E}, \\ \delta \Delta &= \cos \mathrm{R\,F}\, \delta r + r \cos \mathrm{S\,F}\, (\delta v + \delta \chi_0) + \delta \mathrm{Z} \cos \mathrm{W\,F}, \end{aligned}\right\} \begin{array}{l} \textit{Erg. Heft,} \\ \text{(4), p. 162.} \end{array}$$

In these formulæ we have put (see (20) above) our δ Z for its equivalent,

$$r \sin (v + \omega) \sin \delta i - r \cos (v + \omega) \sin i\, \delta \Omega,$$

which is a distance in an absolutely determined direction, and must therefore be the same if we employ ω_0, i_0, Ω_0, instead of ω, i, Ω. In general, however, the plane (I.) will be the ecliptic. If we transform the above-cited formulæ, as Goetze has done, into the following,

$$\cos \mathrm{R\,D} = \sin \mathrm{W\,D} \sin \mathrm{M}' \qquad \cos \mathrm{R\,E} = \sin \mathrm{W\,E} \sin \mathrm{N}' \qquad \cos \mathrm{R\,F} = \sin \mathrm{W\,F} \sin \mathrm{P}';$$

$$\cos \mathrm{S\,D} = \sin \mathrm{W\,D} \cos \mathrm{M}' \qquad \cos \mathrm{S\,E} = \sin \mathrm{W\,E} \cos \mathrm{N}' \qquad \cos \mathrm{S\,F} = \sin \mathrm{W\,F} \cos \mathrm{P}';$$

and make

$$\begin{aligned} \mathrm{D}_z\, r = \mathrm{D}_t\, r &= q^0 \cos \mathrm{Q}^0, \\ r\, \mathrm{D}_z\, (v + \chi_0) &= q^0 \sin \mathrm{Q}^0, \end{aligned} \qquad \text{whence,} \begin{cases} q^0 \cos \mathrm{Q}^0 = \dfrac{k}{\sqrt{p}}\, e \sin v = \dfrac{k \sqrt{p}}{r} \tan \varphi \sin \mathrm{E}; \\ q^0 \sin \mathrm{Q}^0 = \dfrac{k \sqrt{p}}{r}; \end{cases}$$

we shall have, by similar equations to his (11), (12), in the article above cited,

$$\left.\begin{aligned} \cos b\, \mathrm{D}_z\, l &= \frac{q^0}{\Delta} \sin \mathrm{W\,D} \sin (\mathrm{M}' + \mathrm{Q}^0), \\ \mathrm{D}_z\, b &= \frac{q^0}{\Delta} \sin \mathrm{W\,E} \sin (\mathrm{N}' + \mathrm{Q}^0), \\ \mathrm{D}_z\, \Delta &= q^0 \sin \mathrm{W\,F} \sin (\mathrm{P}' + \mathrm{Q}^0). \end{aligned}\right\} \tag{21}$$

Again, as

$$\mathrm{D}_w\, r = r,$$

$$\mathrm{D}_w\, v = 0,$$

the equations before quoted give us,

$$\left.\begin{aligned} \cos b\, D_w\, l &= \frac{r}{\Delta} \sin W D \sin M'; \\ D_w\, b &= \frac{r}{\Delta} \sin W E \sin N'; \\ D_w\, \Delta &= r \sin W F \sin P'. \end{aligned}\right\} \quad (22)$$

And, finally,

$$\left.\begin{aligned} \cos b\, D_Z\, l &= \frac{\cos W D}{\Delta}; \\ D_Z\, b &= \frac{\cos W E}{\Delta}; \\ D_Z\, \Delta &= \cos W F. \end{aligned}\right\} \quad (23)$$

The following are (*Astr. Nachr.* XXVIII. 115) Goetze's general expressions, by means of which W D, M', &c. may be computed:

$$\left.\begin{aligned} \sin W D \sin M &= -\sin (l - \Omega'); \\ \sin W D \cos M &= \cos (l - \Omega') \cos I; \\ \cos W D &= -\cos (l - \Omega') \sin I; \\ M' &= M + \omega' + v; \end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned} \sin W E \sin N &= -\cos (l - \Omega') \sin b; \\ \sin W E \cos N &= -\sin (l - \Omega') \sin b \cos I + \cos b \sin I; \\ \cos W E &= \sin (l - \Omega') \sin b \sin I + \cos b \cos I; \\ N' &= N + \omega' + v; \end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned} \sin W F \sin P &= \cos (l - \Omega') \cos b; \\ \sin W F \cos P &= \sin (l - \Omega') \cos b \cos I + \sin b \sin I; \\ \cos W F &= -\sin (l - \Omega') \cos b \sin I + \sin b \cos I; \\ P' &= P + \omega' + v. \end{aligned}\right\} \quad (26)$$

In these expressions, Ω', ω', I, denote respectively the distance of the planet's ascending node upon the plane II., counted from the same origin as l is, the distance (in the direction of motion) from this ascending node to the perihelion, and the inclination to the plane II.

15. We will now let the approximate orbit-plane itself become the plane II. We have first to devise means for referring observed geocentric places to it. We shall, therefore, suppose that the planet's geocentric orbit-longitude, so to speak, is denoted by η, it being counted from the ascending node upon the equator; that θ denotes, likewise, its geocentric orbit-latitude, referred in the same way to the orbit.

It must be observed that throughout this portion of the investigation we are speaking of the approximate orbit as a *fixed* plane of reference.

Let Ω_1, ω_1, i_1, denote the approximate longitude of the ascending node upon the equator,

the distance (in the direction of the motion) of the perihelion from this node, and the inclination to the equator. Let α, δ denote the geocentric right-ascension and declination.

Then will η, θ be the same functions of $\alpha - \Omega_1$, δ, i_1, that the geocentric longitude and latitude are of α, δ, and the obliquity.

We shall thus have (*Theoria Motus*, Art. 68, p. 64),

$$\left.\begin{aligned}
\sin(45^\circ - \tfrac{1}{2}\theta)\sin\tfrac{1}{2}(\mathrm{E}-\eta) &= \cos[45^\circ + \tfrac{1}{2}(\alpha - \Omega_1)]\sin[45^\circ - \tfrac{1}{2}(i_1 + \delta)];\\
\sin(45^\circ - \tfrac{1}{2}\theta)\cos\tfrac{1}{2}(\mathrm{E}-\eta) &= \sin[45^\circ + \tfrac{1}{2}(\alpha - \Omega_1)]\cos[45^\circ - \tfrac{1}{2}(i_1 - \delta)];\\
\cos(45^\circ - \tfrac{1}{2}\theta)\sin\tfrac{1}{2}(\mathrm{E}+\eta) &= \sin[45^\circ + \tfrac{1}{2}(\alpha - \Omega_1)]\sin[45^\circ - \tfrac{1}{2}(i_1 - \delta)];\\
\cos(45^\circ - \tfrac{1}{2}\theta)\cos\tfrac{1}{2}(\mathrm{E}+\eta) &= \cos[45^\circ + \tfrac{1}{2}(\alpha - \Omega_1)]\cos[45^\circ - \tfrac{1}{2}(i_1 + \delta)].
\end{aligned}\right\} \quad (27)$$

It will not be necessary in general to compute η, θ with extreme accuracy, as what we most wish are their variations. These will be (see same article in *Theoria Motus*, *ad finem*, where, as in (27), E is *not* the eccentric anomaly),

$$\left.\begin{aligned}
d\,\eta\cos\theta &= \sin\mathrm{E}\,d\,\alpha\cos\delta + \cos\mathrm{E}\,d\,\delta;\\
d\,\theta &= -\cos\mathrm{E}\,d\,\alpha\cos\delta + \sin\mathrm{E}\,d\,\delta.
\end{aligned}\right\} \quad (27')$$

In applying now the formulæ (24) to (26), we shall have $\mathrm{I} = 0$. Ω', indefinite, but $\Omega' + \omega' = \omega$, as the origin of η is at the ascending node, on the equator. The letters η and θ are substituted, as the things which they denote are here the same, for l and b, Goetze's notation.

Then,

$$\left.\begin{aligned}
\mathrm{W\,D} &= 90^\circ;\\
\mathrm{M} &= -\eta + \Omega';\\
\mathrm{M}' &= \omega_1 + v - \eta;
\end{aligned}\right\} \quad (24')$$

$$\left.\begin{aligned}
\mathrm{W\,E} &= \theta;\\
\mathrm{N}' &= 270^\circ + \omega_1 + v - \eta;
\end{aligned}\right\} \quad (25') \qquad \left.\begin{aligned}
\mathrm{W\,F} &= 90^\circ - \theta;\\
\mathrm{P}' &= 90^\circ + \omega_1 + v - \eta.
\end{aligned}\right\} \quad (26)$$

From (23), (24′) we have the expression

$$\mathrm{D}z\,\eta = 0; \quad (27)$$

and as the variations z, w do not contain Ω or i, and are thus (to terms of the first order with respect to the changes of the elements) independent of small changes in the orbit-plane, so also is η.

That is, so soon as we know the positions of the orbit-plane correct within limits of error of the first order, we can obtain *one* geocentric coördinate which depends only upon four elements.

16. If a set of rectangular coördinates be employed, the axes of x, y in which are in the approximate orbit-plane, and the former passes through the nodes upon the equator, the positive direction of x being towards the ascending node, and if Ξ, Υ, Ψ denote the sun's geocentric coördinates, we shall have, if the elements are supposed unvaried (Δ denoting, as usual, geocentric distance),

$$\left.\begin{aligned} r^\circ \cos(\omega_1{}^\circ + v^\circ) + \Xi &= \varDelta^\circ \cos\eta^\circ \cos\theta^\circ ; \\ r^\circ \sin(\omega_1{}^\circ + v^\circ) + \Upsilon &= \varDelta^\circ \sin\eta^\circ \sin\theta^\circ ; \\ \Psi &= \varDelta^\circ \sin\theta^\circ . \end{aligned}\right\} \tag{28}$$

The elements being supposed changed, we shall have $\delta(\omega_1 + v) = \delta\, A$, provided we consider $\delta\,\omega_1$ as the variation arising from a change in the position of the perihelion, and do not permit it to be affected by the change in the orbit-plane.

So that Q being any angle whatever, and Ξ, Υ, Ψ being invariable, we shall have

$$\delta\,[r \sin(\omega_1 + v + \mathrm{Q})] = \delta\,[\varDelta \sin(\eta + \mathrm{Q}) \cos\theta]. \tag{28'}$$

17. We will now apply the formulæ we have obtained to a practical problem of very frequent occurrence, — the case in which a set of elements is desired which shall satisfy two observed or normal positions precisely, and certain others nearly.

In our case we will, with regard to those positions which are not to be exactly represented, content ourselves with satisfying the observed value of our coördinate, η; the plane to which it refers being that of an approximate orbit satisfying the first-mentioned positions.

We will, therefore, assume values of $\varDelta$ corresponding to times t_1, t_2, for which we have observed right-ascensions. From this the orbit is easily found, corresponding to these distances.

A variation of the distance for the time t, which we may denote by $\varDelta$, will produce the following change in equation (28):

$$\delta\,[r \sin(\omega_1 + v + \mathrm{Q})] = \delta\,\varDelta \,.\, \sin(\eta + \mathrm{Q}) \cos\theta. \tag{29}$$

But we put $\delta\,\mathrm{L}_{,} = \delta(\omega_1 + v)$ under the form $\delta\, z \,.\, \mathrm{D}_t\, v = \delta\, z \,.\, \mathrm{D}_t\, A$, as in (9); whence by (18)

$$\delta \log r = \mathrm{D}_t \log r \,.\, \delta\, z + r\, \delta\, w,$$

and (29) becomes

$$\delta\,[r \sin(\omega_1 + v + \mathrm{Q})] = \mathrm{D}_t\,[r \sin(\omega_1 + v + \mathrm{Q})]\, \delta\, z + r \sin(\omega_1 + v + \mathrm{Q})\, \delta\, w.$$

Making now

$$\left.\begin{aligned} \mathrm{D}_t\, r &= \frac{k\sqrt{p}}{r} \tan\varphi \sin \mathrm{E} = c\cos\psi, \\ r\, \mathrm{D}_t\, v &= \frac{k\sqrt{p}}{r} = c \sin\psi, \end{aligned}\right\} \tag{30}$$

we shall have, if δ denote the effect of changing $\varDelta$ only,

$$\delta\,\varDelta \sin(\eta + \mathrm{Q}) \cos\theta = c \sin(\omega_1 + v + \psi + \mathrm{Q})\, \delta\, z + r \sin(\omega_1 + v + \mathrm{Q})\, \delta\, w. \tag{31}$$

If $\mathrm{Q} = -\omega_1 - v - \psi$, (31) becomes

$$\delta\,\varDelta \sin(\eta - \omega_1 - v - \psi) \cos\theta = -r \sin\psi \,.\, \delta\, w;$$

$$\mathrm{D}_\Delta\, w = \frac{\sin(\omega_1 + v + \psi - \eta)}{r \sin\psi} \cos\theta. \tag{32}$$

So also, making $Q = -\omega_1 - v$, we find

$$D_\Delta z = \frac{\sin(\eta - \omega_1 - v)}{c \sin\psi} \cos\theta = \frac{r \sin(\eta - \omega_1 - v)}{k\sqrt{p}} \cos\theta. \tag{33}$$

If $\varDelta$, and consequently the orbit, is changed, so that the plane to which the coördinates are referred is no longer the actual plane of the orbit, the last equation of the group (27) becomes

$$\delta Z + \Psi = \varDelta \sin\theta. \tag{34}$$

And thus

$$D_\Delta Z = \sin\theta.$$

An arbitrary variation in either assumed geocentric distance is thus separated into three components; one, $c\,\delta z$, in the direction of the tangent to the orbit, at the planet's place for the time to which the varied distance refers, c being the tangential velocity; another, $r\,\delta w$, in the direction of the radius vector, and so not making an angle $= 90°$, but one $= \psi$, with the former; and a third δZ, perpendicular to the orbit-plane, which is the plane in which the other two lie.

A new orbit computed with the distance $\varDelta_1$, varied by the amount $\delta\varDelta_1$, and the distance $\varDelta_2$ unvaried, but still using $\alpha_1, \alpha_2, \delta_1, \delta_2$, will be connected with the former in the following way.

If $f_1 = D_{\Delta_1} z_1$, $g_1 = D_{\Delta_1} w_1$, then there will be four equations, of which the known terms will be f_1, $D_{\Delta_1} z_2 = 0$, g_1, $D_{\Delta_1} w_2 = 0$, and the unknown the differential coefficients of these elements which lie in the orbit-plane; the coefficients of these unknown quantities being the coefficients of the variations of the elements in (11) and (19′).

But instead of μ as an element, we shall use p, whereby

$$d\log\mu = -\tfrac{3}{2}\, d\log p + 3\, d\log\cos\varphi\,;\ \text{or,}$$
$$d\log\mu = -\tfrac{3}{2}\, d\log p - 3\tan\varphi\, d\varphi. \tag{13}$$

Our four equations will be, then,

$$\left.\begin{aligned}
f_1 &= \frac{1}{\mu} D_{\Delta_1} L_{,} - \tfrac{3}{2} t_1 D_{\Delta_1} \log p + \left(\frac{r_1 + p}{\mu\, a\cos\varphi} \sin E_1 - 3\tan\varphi \,.\, t_1\right) D_{\Delta_1}\varphi + \frac{1}{\mu}\left(\frac{r^2}{a^2\cos\varphi} - 1\right) D_{\Delta_1}\chi\,;\\
0 &= \frac{1}{\mu} D_{\Delta_1} L_{,} - \tfrac{3}{2} t_2 D_{\Delta_1} \log p + \left(\frac{r_2 + p}{\mu\, a\cos\varphi} \sin E_2 - 3\tan\varphi \,.\, t_2\right) D_{\Delta_1}\varphi + \frac{1}{\mu}\left(\frac{r^2}{a^2\cos\varphi} - 1\right) D_{\Delta_1}\chi\,;\\
g_1 &= \qquad D_{\Delta_1}\log p - \frac{r_1\cos v_1}{a\cos\varphi} D_{\Delta_1}\varphi - \tan\varphi \sin E_1\, D_{\Delta_1}\chi\,;\\
0 &= \qquad D_{\Delta_1}\log p - \frac{r_2\cos v_2}{a\cos\varphi} D_{\Delta_2}\varphi - \tan\varphi \sin E_2\, D_{\Delta_1}\chi.
\end{aligned}\right\} \tag{35}$$

By subtracting the first of these equations from the second, we can at once reduce our equations to the number of three, with three unknown quantities; determining $D_{\Delta_1} L_{,}$ afterwards from the second equation explicitly, or, what is perhaps better, leaving $\frac{1}{\mu} \,.\, D_{\Delta_1} L_{,}$ expressed in terms of the variations $D_{\Delta_1}\log p$, &c., in which form it can be easily used.

If now we vary Δ_2, we shall have similar equations, in which, after performing the subtraction indicated above, we shall have, in place respectively of $-f_1, g_1, 0$, as known terms, the values $f_2, 0, g_2$; and as unknown quantities, the first derivatives of the elements with respect to Δ_2.

18. Having now these derivatives with respect both to Δ_1 and Δ_2, we can for any other time easily compute, by formulæ (11), (19), using the former as modified by (13),

$$\delta z = \frac{1}{\mu} \delta L_{,} - \tfrac{3}{2} t \, \delta \log p + \left(\frac{r+p}{\mu \, a \cos \varphi} \sin E - 3 t . \tan \varphi \right) \delta \varphi + \frac{1}{\mu} \left(\frac{r^2}{a^2 \cos \varphi} - 1 \right) \delta \chi, \tag{36}$$

the values of $D_{\Delta_1} z$, $D_{\Delta_1} w$; and as, by (26′),

$$D_Z \eta = 0,$$

the value (which we do not know) of δZ will not affect η; and the coefficients $D_{\Delta_1} \theta$, $D_{\Delta_2} \theta$ are, with regard to almost all planets, of such slight amount that they cannot increase the weight of our solution much over $\frac{1}{100}$ part, as has been shown in Mr. G. P. Bond's Memoir on "Equivalent Factors," assuming the modulus of $D_{\Delta_1} \theta$ as about $\frac{1}{10}$, which is as much as it can often be with regard to five sixths of the asteroids.

19. We have thus, if at least two positions besides those to be exactly satisfied are given, the means of determining what changes the assumed Δ_1, Δ_2 must undergo to satisfy all the geocentric places we have, with almost if not quite the accuracy of a least-square solution. After this is done, it only remains to compute from the variations of Δ_1, Δ_2 the changes which the orbit-plane must undergo, in order to pass exactly through the positions to which they refer. This is done by combining (20) and (34).

We have

$$\left.\begin{aligned} \delta Z_1 &= \sin \theta_1 \, \delta \Delta_1 = r_1 \sin (v_1 + \omega) . \delta i - r_1 \sin i \cos (v_1 + \omega) . \delta \Omega; \\ \delta Z_2 &= \sin \theta_2 \, \delta \Delta_2 = r_2 \sin (v_2 + \omega) . \delta i - r_2 \sin i \cos (v_2 + \omega) . \delta \Omega. \end{aligned}\right\} \tag{37}$$

The readiest practical way to obtain δi, $\delta \Omega$ will be to form these equations and solve them numerically.

After this is done, we shall find also, by the equations in which D_{Δ_1}, $L_{,}$, &c. occurred, the values of $\delta L_{,}$, $\delta \log p$, $\delta \varphi$, $\delta \chi$; and the resulting changes in the usual elements will be,

$$\left.\begin{aligned} \delta L &= \delta L_{,} + (1 - \cos i) \, \delta \Omega; \\ \delta \pi &= \delta \chi + (1 - \cos i) \, \delta \Omega; \\ \delta \log a &= \delta \log p + 2 \tan \varphi . \delta \varphi \sin 1''; \\ \delta \mu &= - \tfrac{3}{2} \mu \, \delta \log p - 3 \mu \tan \varphi . \delta \varphi \sin 1''. \end{aligned}\right\} \tag{38}$$

It is to be noticed that all the variations, those of Δ_1, Δ_2, and log p inclusive, will be expressed in seconds; so that we must add to their logarithms the log 4.6855749 — 10, or must divide them by 206264.8 to reduce them to the unit required.

20. As an example, we have selected the planet Aglaja (47). But few observations of it were made near its first observed opposition, in 1857; and five normals contain as much of the material at hand, as, on account of its scattered nature, we have thought it well to use. For several very exact normals do not help the matter much, if they must, from the nature of the case, be so combined with less exact ones that the burden shall fall on these last; the chain is not stronger than its weakest link.

The normals are: —

Number.	Date. Washington M. T.	No. Obs.	α	δ
			° ′ ″	° ′ ″
I.	1857, Sept. 17.5	13	0 21 29.8	—0 49 44.2
II.	" Oct. 21.0	11	354 21 7.6	—2 11 17.4
III.	" Nov. 16.0	9	353 38 16.4	—1 45 47.6
IV.	1858, Jan. 11.0	3	3 52 10.9	+3 32 41.7
V.	" Feb. 10.5	2	13 21 32.7	+7 52 1.2

Assuming the following for the logarithms of two geocentric distances,

Sept. 17.5 0.201821, Nov. 16.0 0.307251,

the resulting elements, referred to the equator and mean equinox of 1857, Jan. 0, were,

Epoch, 1857, Nov. 16.0	
	° ′ ″
M°	46 20 24.3
ω_1	313 31 56.7
Ω_1	0 49 9.5
i_1	28 27 7.0
φ°	7 21 56.8
μ°	12 5.719

The other normals were represented thus,

Number.	Date. Washington M. T.	C. — O. $\Delta\alpha$	C. — O. $\Delta\delta$
		″	″
II.	1857, Oct. 21.0	+ 0.4	+ 0.1
IV.	1858, Jan. 11.0	+25.4	+11.9
V.	" Feb. 10.5	+54.9	+33.0

For those dates the equations (27), (27′) give us,

	η	θ	E	C. — O.	
	° ′	° ′	° ′	″	″
II.	353 15.58	+1 9.04	61 44.36	$\delta\eta =$ + 0.40	$\delta\theta =$ —0.10
IV.	4 21.78	+1 40.38	61 34.72	+27.96	—1.60
V.	14 44.20	+1 1.88	62 16.84	+63.50	+3.91

For the dates I. and III., we find

	η	θ
	° ′	° ′
I.	359 11.38	—0 30.50
III.	352 49.85	+1 51.75

As we do not wish E here, we can apply a method like the simpler well-known forms of converting right-ascension and declination into latitude and longitude.

From the elements

$$\log (k \sqrt{p^\circ}) = 8.461728\,;$$

and from what has been previously obtained,

	$\log r$	v	ψ
Sept. 17.5	0.413888	$43^\circ\ 50'.67$	$85^\circ\ 21'.16$
Nov. 16.0	0.423901	58 11.80	84 10.36

From (32) and (33) are now derived,

For		$\log f.$	$\log g.$
	$t_1 =$ Sept. 17.5	$+0.45230$	$+9.58475$
	$t_2 =$ Nov. 16.0	-1.47254	$+9.56672$

The equations (36) for determining D_{Δ_1}, $\log p$, &c. become, subtracting the second of them from the first, and also those with the same coefficients for finding $D_{\Delta_2} \log p$, &c. if we make $D_{\Delta_1} \log p = l_1$, $D_{\Delta_2} \log p = l_2$, $D_{\Delta_1} \varphi = m_1$, $D_{\Delta_2} \varphi = m_2$, $D_{\Delta_1} \chi = n_1$, $D_{\Delta_2} \chi = n_2$,

$$\left.\begin{array}{rrcl} \overset{(1.)}{+2.833} & \overset{(2.)}{+29.685} & = & (1.95061)\, l - (1.83514)\, m - (1.03991)\, n\,; \\ +0.38437 & 0.00000 & = & (0.00000)\, l - (9.81607)\, m - (8.91004)\, n\,; \\ 0.00000 & +\ 0.36874 & = & (0.00000)\, l - (9.68983)\, m - (9.00885)\, n\,; \end{array}\right\} \tag{39}$$

of which the solutions are,

$\log (D_{\Delta_1} \log \text{nat}\, p) = +9.8222\,;$ $\quad \log (D_{\Delta_1} \varphi) = -9.9737\,;$ $\quad \log (D_{\Delta_1} \chi) = +1.0423\,;$

$\log (D_{\Delta_2} \log \text{nat}\, p) = -9.4000\,;$ $\quad \log (D_{\Delta_2} \varphi) = +9.9620\,;$ $\quad \log (D_{\Delta_2} \chi) = -1.0199.$

When $\varDelta_1$ is changed, the value of z for t_2 remains unchanged, and *vice versa;* so that, for any time,

$$D_{\Delta_1} z = D_{\Delta_1} (z - z_2)\,; \tag{40}$$

But $z - z_2$ does not contain $L_,$, so that we shall have

$$D_{\Delta_1} z = D_{\log p} (z - z_2)\, D_{\Delta_1} \cos p + D_\varphi\, (z - z_2)\, D_{\Delta_1} \varphi + D_\chi\, (z - z_2)\, D_\Delta\, \chi. \tag{40'}$$

Similarly, $\qquad D_{\Delta_2} z = D_{\Delta_2} (z - z_1).$ (41)

Nor does δw contain $\delta L_,$.

We now find for the times of normals II., IV., and V., by formulæ (11), (19′), as modified in (40), (41),

	$D_{\Delta_1} z$	$D_{\Delta_1} w$	$D_{\Delta_2} z$	$D_{\Delta_2} w$
II.	$-\ 4.373$	$+.1645$	$-\ 11.33$	$+.2108$
IV.	$+37.39$	$-.3285$	$-\ 96.07$	$+.6848$
V.	$+72.52$	$-.4890$	-146.44	$+.8395$

And we thus finally form the equations in which $\delta\eta \cos\theta$ is the known term, $\delta\Delta_1$, $\delta\Delta_2$ the unknown quantities, and $\cos\theta\, D_{\Delta_1}\eta$, $\cos\theta\, D_{\Delta_2}\eta$ the coefficients.

From normals

$$\left.\begin{array}{llll} \text{II.} & 0 = +0''.40 & +0.0249\,\delta\Delta_1 & -0.0036\,\delta\Delta_2 \\ \text{IV.} & \quad 27.96 & \quad 0.0284 & -0.1259 \\ \text{V.} & \quad 63.50 & \quad 0.1145 & -0.2672 \end{array}\right\} \tag{41}$$

From these equations were derived, by least squares, the values

$$\delta\Delta_1 = -6 \sin 1'',$$
$$\delta\Delta_2 = +231.2 \sin 1'',$$

the weights being made proportional to the number of observations.

By means of the previously computed partial differential coefficients, we are now enabled to obtain the values

$$\left.\begin{array}{r} \delta \log \text{nat}\, p = -300.9 \\ \therefore \delta \text{ com} \log p = -130.6 \end{array}\right\} \text{in the sixth place of decimals.}$$

$$\log p = 0.452163,$$
$$\delta\varphi = 3'\ 37''.4,$$
$$\varphi = 7^\circ\ 25'\ 34''.2, \quad \text{hence,} \quad \log\cos\varphi\ 9.996342.$$
$$\delta\chi = \chi - \pi^\circ = -41'\ 26''.6,$$
$$\log a = 0.459479.$$

Referring the previous, and still unchanged, node and inclination to the mean ecliptic and equinox of 1857.0, we find

$$\begin{array}{lrrr} & {}^\circ & {}' & {}'' \\ \Omega^\circ & 4 & 28 & 34.7, \\ \omega^\circ & 309 & 47 & 34.0, \\ i^\circ & 5 & 0 & 25.6. \end{array}$$

Hence, $\chi = 313\ 34\ 42.1.$

From the value of $D_{\Delta_2} z_2 = f_2$, we obtain for 1857, Nov. 16.0,

$$\delta z_2 = f_2 \cdot \delta\Delta_2 = -0^d.03328.$$

Consequently, $\delta v_2 = \delta\Delta_2 - \delta\chi = \dfrac{k\sqrt{p}}{r^2_2 \sin 1''}\,\delta z_2 - \delta\chi.$

This then gives us $\delta v_2 = -28''.2 - \delta\chi = +40'\ 58''.4.$

Then $v_2 = 58^\circ\ 52'\ 46''.2,$

$M = 46^\circ\ 49'\ 58''.7.$

Check, $r_2 = \dfrac{p}{1 + e\cos v_2} = r^\circ_2\,(1 + \delta w_2),$

which comes out exactly 0.424080 in both.

From the equations (38) the values of $\delta i = +11''.2$, $\delta\Omega = -16''.1$, were rather roughly estimated; whence $\pi - \chi = 2\sin^2 \frac{1}{2} i \,.\, \delta\Omega = -0''.1$.

The elements will then be,

Epoch, 1857, Nov. 16.0. — M. Equinox, 1857.0.

M	$46^\circ\ 49'\ 58''.7$
π	$313\ 34\ 42.0$
Ω	$4\ 28\ 18.6$
φ	$7\ 25\ 34.2$
i	$5\ 0\ 36.8$
μ	$12\ 5.752$
log a	0.459479.

The remaining errors, $\delta\ \eta \cos \theta$, after the solution of the conditional equations (41), were,

$$\left.\begin{array}{ll} & \text{C.} - \text{O.} \\ \text{II.} & -0''.6 \\ \text{IV.} & -1''.3 \\ \text{V.} & +1''.0 \end{array}\right\} \qquad (42)$$

Substituting the new elements in the formulæ for the geocentric places, we obtain values of $\delta\ \alpha$, $\delta\ \delta$ corresponding with them; and from these, by (27), values of $\delta\ \eta \cos \theta$, differing from those in (42) very slightly.

The resulting differences in the other geocentric coördinate, θ, are nearly as follows:

	$\delta\ \theta$ (C. — O.)
II.	$+1''.3$
IV.	$-2''.0$
V.	$+5''.7$;

which is a small difference, seeing that the last normal depends upon but two observations, and those, as is apparent by Mr. Oeltzen's investigation (*Astr. Nachr.*, No. 1167) disagreeing largely, as indeed do the observations upon which IV. is based.

Ten months after the date of the last of these normals, the comparison of an ephemeris computed from the above elements with a Cambridge observation, kindly furnished me by Prof. W. C. Bond, was as follows:

	C. — O.	
	$\Delta\ \alpha \cos \delta$	$\Delta\ \delta$
1858, Dec. 28	$+1'.6$	$+1'.2$;

showing an agreement much more satisfactory than is usual in determinations from the first apparition of an asteroid, especially considering the weakness of normals IV. and V., upon which, as may be seen by (41), almost the entire weight of the computation rests. Perturbations, too, were entirely neglected.

The calculation of the above orbit was executed entirely with six decimal places (less

in portions). The usual method (of varying two mean distances, actually computing three orbits) would not only have required more operations, but these must, many of them, have been carried to seven decimals, in order to obtain with any degree of accuracy the coefficients of the unknown quantities in the equations (41). At least one third the labor appears to be saved by this process, while a differential method, in general much better than the rule of false position, is substituted for the latter.

Moreover, the general relations developed in the first part of this paper are not without very interesting aspects with regard to the general problem of representing long series of geocentric observations by an orbit, by the method of least squares; and it is not impossible that further calculations may be made upon the theory of Aglaja, using the numbers and formulæ already obtained, and approximating more and more closely to the true elements, when the observations requisite shall have been made.